SOME EFFECTS OF THE BIAXIAL TENSION OF AN ELASTIC PLATE WITH A CRACK

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This article examines the elastic equilibrium of a circular plate with a central crack when radial displacements distributed in accordance with an elliptic law are assigned on the boundary of the plate. The problem is reduced to a singular integral equation. Numerical and analytical solutions are obtained and compared with one another. We determine the dependence of the stress-intensity factor at the crack tip on the biaxial loading parameter. As the latter, we take the ratio of the displacements along the principal axes of deformation of the plate. The possibility of stable crack growth is established, and practical applications of the results are noted.

1. Formulation of the Problem. The formulation of the problem first of all reflects the need to develop a mathematical model to interpret experimental results obtained on special equipment which makes it possible to realize two-dimensional tension of circular plate specimens with prescribed radial displacements. Such an approach is promising from the viewpoint of practical considerations. In fact, if a crack forms in a structure, there is almost always the possibility of measuring the displacement at the boundary of a certain relatively small region surrounding the crack. In connection with this, it is of interest to examine the problem of the theory of elasticity for a circular plate of radius R with a central crack of length 2ℓ, with the radial displacements v_ρ assigned at the boundary of the plate – the circle L_0 (Fig. 1).

2. Integral Equation of the Problem. A problem formulated on the basis of [1] reduces to the solution of a singular integral equation relative to the unknown function g(t), which is proportional to the derivative of the displacements on the edges of the crack:

$$\frac{1}{\pi i}\int_{-l}^{l}\frac{g(t)}{t-t_0}\,dt+\frac{1}{2\pi i}\int_{-l}^{l}S(t_0,t)\,g(t)\,dt=F(t_0),\quad -l<t_0<l \tag{2.1}$$

with the condition

$$\int_{-l}^{l}g(t)\,dt=0,$$

where

$$S(t_0,t)=\frac{4t}{\varkappa(\varkappa-1)R^2}+\frac{1}{\varkappa}\frac{(\varkappa^2+7)t-6t_0}{R^2-tt_0}-\frac{2}{\varkappa}\frac{t(t-t_0)(2t-3t_0)}{(R^2-tt_0)^2}-\frac{2}{\varkappa}\frac{t^2t_0(t-t_0)}{(R^2-tt_0)^3};$$

$$F(t_0)=-\frac{1}{2\pi i}\int_{L_0}\left\{f(t)\left[\frac{1}{\varkappa(\varkappa-1)}\frac{\varkappa t-t_0}{t(t-t_0)}+\frac{t}{R^2-tt_0}\right]+\overline{f(t)}\left[\frac{1}{\varkappa(\varkappa-1)}\frac{\varkappa R^2-tt_0}{t(R^2-tt_0)}-\frac{R^2}{\varkappa}\frac{t-t_0}{(R^2-tt_0)^2}\right]\right\}dt; \tag{2.2}$$

$$f(t)=\frac{2\mu}{R}\left(v_\rho-i\frac{dv_\rho}{d\theta}\right). \tag{2.3}$$

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TABLE 1

n	$k_1\sqrt{R}(\varkappa-1)/[2\mu(v_1+v_2)]$
3	1,023
5	1,023
7	1,023
9	1,023

TABLE 2

b_1	λ = 0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
	$k_1\sqrt{R}(\varkappa-1)/[2\mu(v_1+v_2)]$								
0	0,909	1,20	1,33	1,34	1,28	1,15	0,974	0,755	0,490
0,1	0,844	1,12	1,24	1,26	1,21	1,10	0,952	0,764	0,530
0,2	0,789	1,05	1,17	1,19	1,15	1,06	0,934	0,771	0,564
0,25	0,765	1,02	1,13	1,16	1,12	1,04	0,926	0,774	0,578
0,3	0,743	0,990	1,10	1,13	1,10	1,03	0,918	0,777	0,592
1/3	0,730	0,972	1,09	1,11	1,09	1,02	0,914	0,779	0,600
0,4	0,704	0,939	1,05	1,08	1,06	0,997	0,905	0,782	0,616
0,5	0,670	0,895	1,00	1,04	1,02	0,972	0,893	0,787	0,637
0,6	0,640	0,856	0,963	1,00	0,991	0,950	0,883	0,791	0,655
2/3	0,622	0,833	0,939	0,977	0,972	0,936	0,877	0,793	0,666
0,7	0,614	0,822	0,927	0,967	0,963	0,930	0,874	0,794	0,672
0,8	0,590	0,792	0,895	0,937	0,938	0,913	0,866	0,797	0,686
0,9	0,569	0,765	0,867	0,910	0,916	0,897	0,859	0,800	0,699
1	0,551	0,740	0,841	0,886	0,896	0,883	0,853	0,803	0,711
1,5	0,479	0,648	0,743	0,794	0,820	0,830	0,829	0,812	0,755
2	0,431	0,586	0,677	0,733	0,769	0,794	0,813	0,819	0,784
3	0,372	0,509	0,596	0,657	0,705	0,750	0,792	0,826	0,821
4	0,336	0,462	0,547	0,611	0,667	0,723	0,780	0,831	0,843
5	0,312	0,431	0,514	0,580	0,642	0,706	0,772	0,834	0,857
6	0,295	0,409	0,491	0,559	0,624	0,693	0,766	0,837	0,868
7	0,282	0,393	0,473	0,542	0,610	0,683	0,762	0,838	0,876
8	0,272	0,380	0,460	0,530	0,600	0,676	0,759	0,840	0,882
9	0,264	0,370	0,449	0,519	0,591	0,670	0,756	0,841	0,887
10	0,258	0,361	0,440	0,511	0,584	0,665	0,754	0,842	0,891
∞	0,193	0,277	0,351	0,428	0,515	0,617	0,732	0,850	0,931

TABLE 3

λ	$k_1\sqrt{R}(\varkappa-1)/[2\mu(v_1+v_2)]$
0,1	0,670
0,2	0,894
0,3	0,999
0,4	1,02
0,5	0,996
0,6	0,936

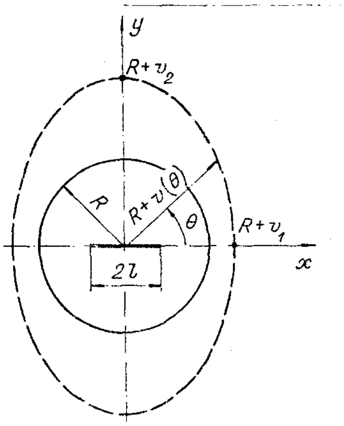

Fig. 1

Here, μ is the shear modulus; $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for a generalized plane stress state; ν is the Poisson's ratio.

We choose the following as the distribution law for the radial displacements on the boundary of the circle

$$v_\rho(\theta) = v_1 \cos^2\theta + v_2 \sin^2\theta \tag{2.4}$$

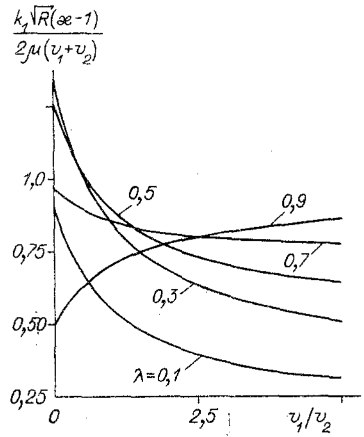

Fig. 2

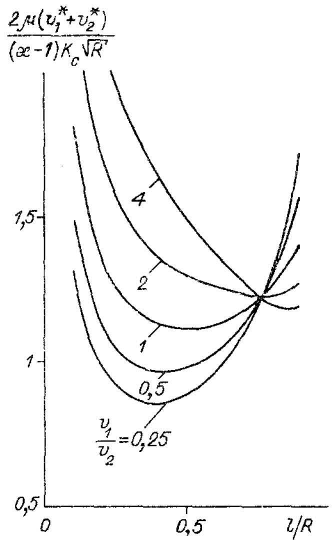

Fig. 3

(v_1 and v_2 are the radial displacements along the x- and y-axes, respectively (see Fig. 1). This choice of displacements leads to a situation whereby the deformed contour of the plate acquires a shape which is close to elliptical for small v_1 and v_2.

With allowance for (2.4), function (2.3) takes the form

$$f(\theta)=\frac{2\mu}{R}\frac{v_1+v_2}{2}(1+b\cos 2\theta+2ib\sin 2\theta), \tag{2.5}$$

where $b=(v_1-v_2)/(v_1+v_2)$ or $b=(b_1-1)/(b_1+1)$; $b_1=v_1/v_2$. Inserting (2.5) into (2.2) and integrating, we find that

$$F(t_0)=-\frac{2\mu}{R}\frac{v_1+v_2}{2}\left(\frac{2}{\varkappa-1}-\frac{b}{2}-\frac{3b}{2\varkappa}+\frac{6b}{\varkappa R^2}t_0^2\right). \tag{2.6}$$

3. <u>Numerical Solution.</u> We obtain the solution of singular integral equation (2.1) by the numerical method proposed in [2]. Considering that $g(t) = -g(-t)$ by virtue of the symmetry of the problem, we write equation (2.1) for the range of integration $[0, \ell]$ as

$$\frac{1}{\pi i}\int_0^l \frac{2t}{t^2-t_0^2}g(t)\,dt+\frac{1}{2\pi i}\int_0^l [S(t_0,t)-S(t_0,-t)]g(t)\,dt=F(t_0),$$
$$0<t_0<l. \tag{3.1}$$

Making the substitution of variables $t=l\tau,\ t_0=l\tau_0,\ 0<\tau,\ \tau_0<1$, in (3.1), we have

$$\int_0^1\left\{\frac{2\tau}{\tau^2-\tau_0^2}+\frac{1}{2}[M(\tau_0,\tau)-M(\tau_0,-\tau)]\right\}g(t)\,d\tau=\pi iL(\tau_0),$$
$$M(\tau_0,\tau)=$$
$$=\frac{\lambda^2}{\varkappa}\left[\frac{4\tau}{\varkappa-1}+\frac{(\varkappa^2+7)\tau-6\tau_0}{1-\lambda^2\tau\tau_0}-2\lambda^2\frac{\tau(\tau-\tau_0)(2\tau-3\tau_0)}{(1-\lambda^2\tau\tau_0)^2}-2\lambda^4\frac{\tau^2\tau_0(\tau-\tau_0)^2}{(1-\lambda^2\tau\tau_0)^3}\right], \tag{3.2}$$
$$L(\tau_0)=-\frac{2\mu}{R}\frac{v_1+v_2}{2}\left[\frac{2}{\varkappa-1}-\frac{b}{2\varkappa}(\varkappa+3-12\lambda^2\tau_0^2)\right],\quad \lambda=\frac{l}{R}.$$

The value $\tau = 1$ in Eq. (3.2) corresponds to the position of the crack tip, while $\tau = 0$ corresponds to the center of the crack. It is not hard to show that the unknown function g(t) is purely imaginary for the given problem. We represent it in the form

$$g(t) = -\frac{2\mu(v_1+v_2)}{(\varkappa-1)R} i\sqrt{\frac{\tau}{1-\tau}}\, u(\tau) \tag{3.3}$$

($u(\tau)$ is a new unknown function). Inserting (3.3) into (3.2) and following the procedures described in [2], we approximate singular integral equation (3.2) by a system of linear algebraic equations relative to $u_m \approx u(\tau_m)$:

$$\begin{aligned}\sum_{m=1}^{n} A_m u_m \left\{\frac{2\tau_m}{\tau_m^2-\tau_{0k}^2} + \frac{1}{2}\left[M(\tau_{0k}, \tau_m) - M(\tau_{0k}, -\tau_m)\right]\right\} = \\ = \pi\left[1 - b\,\frac{\varkappa-1}{4\varkappa}\left(\varkappa+3-12\lambda^2\tau_{0k}^2\right)\right], \quad k = 1, 2, \ldots, n.\end{aligned} \tag{3.4}$$

Here,

$$\tau_m = \sin^2\frac{m\pi}{2n}, \quad m = 1, 2, \ldots, n; \quad \tau_{0k} = \sin^2\frac{2k-1}{4n}\pi, \quad k = 1, 2, \ldots, n;$$

$$A_m = \frac{\pi}{n}\sin^2\frac{m\pi}{2n}, \quad m = 1, 2, \ldots, n-1; \quad A_n = \frac{\pi}{2n}.$$

Using formulas from [3] which connect the function g(t) with the stress-intensity factor at the crack tip k_1, we find that

$$\frac{k_1\sqrt{R}(\varkappa-1)}{2\mu(v_1+v_2)} = \sqrt{2\pi\lambda}\, u(1). \tag{3.5}$$

The value $u(1) \approx u_n$ entering into (3.5) is determined directly from the solution of system (3.4). This is one of the advantages of the the numerical approach we are using.

4. Approximate Analytical Solution. Together with a numerical solution, we obtain an approximate analytical solution to Eq. (2.1) with small values of $\lambda = \ell/R$ (the ratio of the half-length of the crack to the radius of the circle). We keep terms whose order does not exceed λ^2 in the expansion of the kernel of the equation in powers of λ, and we refer the length to ℓ in (2.1). We then obtain the approximate equation

$$\frac{1}{\pi i}\int_{-1}^{1}\frac{g(t)}{t-t_0}\,dt + \frac{\lambda^2}{2\pi i}\int_{-1}^{1}\left(4\eta t - \frac{6t_0}{\varkappa}\right)g(t)\,dt = F(t_0), \quad -1 < t_0 < 1 \tag{4.1}$$

with the condition

$$\int_{-1}^{1} g(t)\,dt = 0, \tag{4.2}$$

where $F(t_0)$ is determined by Eq. (2.6); $\eta = (\varkappa^3 - \varkappa^2 + 7\varkappa - 3)/[4\varkappa(\varkappa-1)]$. We introduce the notation

$$\int_{-1}^{1} t g(t)\,dt = A = \text{const}. \tag{4.3}$$

Then, with allowance for (4.2), Eq. (4.1) becomes

$$\int_{-1}^{1}\frac{g(t)}{t-t_0}\,dt = \pi i F(t_0) - 2\eta\lambda^2 A. \tag{4.4}$$

We will seek the unknown function g(t) in the class of functions unbounded on the ends of the range of integration. With condition (4.2), the solution of Eq. (4.4) is written as [4]

$$g(t_0) = -\frac{1}{\pi^2\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,[\pi i F(t) - 2\eta\lambda^2 A]}{t-t_0}\,dt. \tag{4.5}$$

Having completed integration in (4.5), we have

$$g(t)=\frac{t}{\sqrt{1-t^2}}\left[i\alpha\left(\frac{\gamma}{2}-c-\gamma t^2\right)-\frac{2\eta\lambda^2}{\pi}A\right]. \qquad (4.6)$$

Here, $\alpha=\frac{2\mu}{R}\frac{v_1+v_2}{2}$; $c=\frac{2}{\varkappa-1}-\frac{b}{2}-\frac{3b}{2\varkappa}$; $\gamma=\frac{6b\lambda^2}{\varkappa}$. Inserting (4.6) into (4.3), we arrive at an equation relative to the unknown A:

$$\int_{-1}^{1}\frac{t^2}{\sqrt{1-t^2}}\left[i\alpha\left(\frac{\gamma}{2}-c+\gamma t^2\right)-\frac{2\eta\lambda^2}{\pi}A\right]dt=A. \qquad (4.7)$$

Solving (4.7), we have $A=-\alpha\pi i(c+\gamma/4)/[2(1+\eta\lambda^2)]$, and, using (4.6), we obtain

$$g(t)=\frac{it}{\sqrt{1-t^2}}\frac{\alpha}{2(1+\eta\lambda^2)}[\gamma(1-2t^2)-2c] \qquad (4.8)$$

(terms whose order is no greater than λ^2 are kept in the bracket).

If we use formulas found in [3] to connect the function g(t) with the stress-intensity factor at the crack tip k_1, we find that

$$\frac{k_1\sqrt{R}(\varkappa-1)}{2\mu(v_1+v_2)}=\frac{\sqrt{\pi\lambda}}{1+\eta\lambda^2}\left[1-b\frac{\varkappa-1}{4\varkappa}(\varkappa+3-6\lambda^2)\right]. \qquad (4.9)$$

5. Numerical Results. Calculations were performed in accordance with the algorithm described in Part 3 for $\varkappa=(3-\nu)/(1+\nu)$, $\nu=0.3$. The error, no greater than 0.1% in any of the cases, was checked by comparing the results of calculations performed with different degrees of accuracy of algebraic approximation of the integral equation – which was determined by the order n of corresponding system (3.4). For example, data is shown in Table 1 for $b_1=v_1/v_2=0.5$, $\lambda=l/R=0.5$. Thus, use of the given numerical method ensures very rapid convergence of the results.

Figure 2 and Table 2 show the dependence of the dimensionless stress-intensity factor on the biaxial-loading parameter $b_1 = v_1/v_2$ for different relative crack dimensions. It should be noted that that the minimum dependence of the dimensionless stress-intensity factor on the parameter b_1 is seen at $\lambda = \ell/R = 0.77$. In this case, with a change in b_1 from 0 to ∞, the quantity $k_1\sqrt{R}(\varkappa-1)/[2\mu(v_1+v_2)]$ changes from 0.825 to 0.816.

Figure 3 shows the dependence of the limiting displacements on relative crack size for different values of b_1. Here, v_1^*, v_2^* and K_c are the critical values of v_1, v_2, and k_1. It can be seen that with an increase in b_1 the tendency for stable crack growth to occur decreases. Table 3 shows results of calculations performed by Eq. (4.9) for $\kappa = (3-\nu)/(1+\nu)$, $\nu = 0.3$, and $b_1 = 0.5$. Comparison of this data with the numerical results in Table 2 shows that Eq. (4.9) is satisfactorily accurate at $\lambda < 0.5$.

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